MATH 200

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SPMI

Claim: $b_n = \dfrac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$ for every integer $n \geq 2$.

Proof: By Strong Principle of Mathematical Induction,

When $n=1$, $\dfrac{(1+\sqrt{2})^1 - (1-\sqrt{2})^1}{2\sqrt{2}} = \dfrac{2\sqrt{2}}{2\sqrt{2}} = 1$, so

$$b_i = \frac{(1+\sqrt{2})^i - (1-\sqrt{2})^i}{2\sqrt{2}}$$

for every integer i with $1 \leq i \leq k$

Assume for a positive integer k,

$$b_{k+1} = \frac{(1+\sqrt{2})^{k+1} - (1-\sqrt{2})^{k+1}}{2\sqrt{2}} \quad \checkmark$$

When $k=1$, $\dfrac{(1+\sqrt{2})^2 - (1-\sqrt{2})^2}{2\sqrt{2}} = \dfrac{4\sqrt{2}}{2\sqrt{2}} = 2 = b_2$, so we can assume $k \geq 2$.

By definition, $b_{k+1} = 2b_k + b_{k-2} = 2\left[\dfrac{(1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}}\right] + \dfrac{(1+\sqrt{2})^{k-1} - (1-\sqrt{2})^{k-1}}{2\sqrt{2}}$

$$= \frac{2(1+\sqrt{2})^k + (1+\sqrt{2})^{k-1} - 2(1-\sqrt{2})^k - (1-\sqrt{2})^{k-1}}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{2})^{k-1}[2(1+\sqrt{2})+1] - (1-\sqrt{2})^{k-1}[2(1-\sqrt{2})+1]}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{2})^{k+1}(1+\sqrt{2})^2 - (1-\sqrt{2})^{k-1}(1-\sqrt{2})^2}{2\sqrt{2}} = \frac{(1+\sqrt{2})^{k+1} - (1-\sqrt{2})^{k+1}}{2\sqrt{2}} \quad \checkmark$$

QED.